A New Development of the Strain Energy Function for Hyperelastic Materials Using a Logarithmic Strain Approach

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ABSTRACT: The development of a new strain energy function for hyperelastic solids based on the logarithmic strain measure is the objective of the present article. For all possible types of deformation it was shown that the proposed energy function is based on three independent material parameters. Using available experimental data for rubber-like materials from the literature, one may determined the materials parameters by a nonlinear fitting. The available domain of the strain energy function can be determined by plotting the third invariant of logarithmic strain vs the second one. The numerical integration of the experimental data of true stress as a function of the logarithmic strain for various types of deformation yields the strain energy function W, for rubber-like solids. The proposed model involves only one parameter that must be determined by fitting with the experimental data. © 2000 John Wiley & Sons, Inc. J Appl Polym Sci 77: 660–672, 2000

Key words: strain energy function; hyperelastic solids; logarithmic strain measure; deformation; material parameters

INTRODUCTION

Rubber-like materials are peculiar in the mechanical and thermodynamic or thermoelastic properties. Early attempts to interpret these properties of rubbers could not overcome the difficulty of explaining its very high extensibility in terms of classical concepts of the structure of matter. The high extensibility coupled with values of Young's modulus of order 1 N mm^{-2} (about 10^{-5} N/mm^2 for normal solids) defied any explanation in terms of cohesional forces between molecules as normally understood. The solution to the problem was more or less coincident with the emergence of the concept of a high polymer—that is, of a material composed of molecules of extremely high molecular weight built up by the successive addition of similar units in the form of a single chain. The acceptance of the concept of a high polymer constituted a revolution of thought not only in relation to the chemistry of these materials, but equally in relation to the interpretation of their physical properties.

Meyer[1] first proposed the connection between the chain-like structure of polymer molecule and the long-standing problem of rubber elasticity. The *kinetic* approach of the rubber elasticity was followed up from the idea of the chain molecule considered as an isolated entity and later of an assembly of chains corresponding to the bulk rubber. This development was due to Guth and Mark[2] and to Kuhn.[3] A variety of theoretical models have been devised for the calculation of the elastic properties of the network.[4–7] All these models embody the same basic physical concepts and lead to substantially similar conclusions.

By a more accurate *statistical* treatment it is possible to derive a relation between the force on a single chain and the distance r between its ends,

which is valid over the whole range of extension.[8] A mathematical theory was also developed by Rivlin,[9,10] which describes the deformation, under the action of applied forces, of bodies of ideal highly elastic materials, which are incompressible and isotropic in their undeformed state. The physical properties of the material are specified in terms of a stored-energy function of two strain invariances I_1 and I_2, which are expressible in terms of the principal extension rations λi ($i = 1$–3) at the point of the deformed body considered. Mooney[11] showed that the form of W is the most general one, which can be valid even for large deformations for an ideal incompressible highly elastic material. The relations between the principal values of stresses and the principal stretches in a pure homogeneous strain involve the partial derivatives of W upon I_1 and I_2. Due to the incompressibility condition for rubbers, only the differences of principal stresses can be determined and not the individual stresses.

From the purely *phenomenological* standpoint, many attempts have been made to obtain a more realistic mathematical formulation of the elastic properties of rubbers than that provided by either the statistical theory or two constant Mooney–Rivlin[9–11] form of stored energy function. In a more general case, the store energy function W was shown to be expressible in the form of even powered series of the principal stretches,[12] which was developed in an analogous way to the Rivlin formulation. Valanis and Ladel[13] proposed a new approach for the estimation of the store energy function of rubber-like materials subjected to various types of deformation . They have proposed as a hypothesis that W must be represented as the sum of separable functions of the three independent extension rations. Additionally, these separable functions must be identical for symmetry considerations. An important development is that of Ogden,[14] who proposed a more general store energy function for rubber materials based on the Valanis–Landel hypothesis. For an incompressible rubber, Ogden expresses W as a series of $\lambda_i^{\alpha_n}$, where α_n can be any real positive or negative number. The proposed W, in conjunction with a good nonlinear fitting algorithm, can fit the experimental data up to break for all types of deformations.

Gent[15] proposed a simple two parameters constitutive relation for W, applicable over the entire range of strains for rubber networks. As an example, experimental data from simple extension deformation were fitted with the proposed store en-

ergy function. However, this function was not applied to all possible types of deformation in order to check the overall validity of the proposed function. As pointed by Gent, the proposed empirical formula has several advantages: (1) it reduces to Neo-Hookean form at small strains, and (2) it is expressed in terms of the strain invariance and as a consequence can be applied to complex states of deformation.

Arruda and Boyce[16] have proposed a constitutive model for the deformation of rubber materials that is shown to represent successfully the response of these materials in uniaxial tension, biaxial extension, uniaxial compression, plane strain compression, and pure shear. The developed constitutive relation is based on an eight-chain representation of the underlying macromolecular network structure of the rubber and the non-Gaussian behavior of the individual chains in the proposed network. The proposed model requires two material parameters, an initial modulus and limiting chain extensibility. Wu and Giessen[17] have developed a three- dimensional molecular network theory, which use a non-Gaussian statistical mechanics model for the large extension of molecules.

An empirical one-parameter equation was also proposed by the author[18] for fitting different types of deformation for rubber-like materials. Using the proposed function with a logarithmic strain uniaxial experimental data, up to high values of strain were fitted. However, the validity of this equation must be checked, and for all the possible type of deformations used in the design of engineering components with rubber. Using a Neo-Hookean strain energy function, the author was able to predict the stress distribution in bonded elastic elastomeric materials.[19]

In the present work, a different way of approaching the problem using the concept of the logarithmic strain is introduced. A new type of strain energy function is implemented in order to fit all possible types of deformation of rubber-like composites. The proposed strain energy function is based on the second and third logarithmic invariant of strain, and involves only three materials parameters. In the section following the nest, preliminary concepts on the kinematics of hyperelasticity are discussed. The logarithmic strain measure and its invariance are discussed in the section after that. The last section shows the form of the strain energy function based on the logarithmic strain measure.

KINEMATICS

This section depicts a brief review of the mathematical aspects of the kinematics of nonlinear elasticity. Let $\underline{\chi}$ denote the motion of a body B; then[19,20]

$$\underline{x} = \underline{\chi}(\underline{X}, t) \tag{1}$$

which defines a family of deformations from the reference configuration. An underline symbolizes a tonsorial notation. In coordinates representation, eq. (1) is written as

$$x^i = \chi^i(X^i, t) \tag{2}$$

The system x^i is related to natural basis e_i and the corresponding X^i is related to basis d_i. We also introduce the corresponding shifter:

$$g_{\kappa}^{\alpha} = e_{\kappa} \cdot d^{\alpha}$$

$$g_{\alpha}^{\kappa} = e^{\kappa} \cdot d_{\alpha} \tag{3}$$

The deformation gradient of the motion χ is a second-order tensor field and is defined as

$$\underline{F} = \nabla\underline{\chi}(\underline{X}, t) \Leftrightarrow F_{\alpha}^{\kappa} = x_{,\alpha}^{\kappa} \equiv \frac{\partial x^{\kappa}}{\partial X_{\alpha}} = \frac{\partial \chi^{\kappa}(X^{\gamma}, t)}{\partial X^{\alpha}} \tag{4}$$

The Jacobean determinant of the deformation gradient describes the transformation (local volumetric deformation) of a referential volume element dV to a volume element dv in the current configuration, i.e.,

$$dv = JdV \Leftrightarrow J \equiv |\det \underline{F}| = \frac{\rho_R}{\rho} \tag{5}$$

where ρ_R and ρ define the densities in the reference and deformed configurations, respectively.

According to polar decomposition theorem,[21,22] any invertible linear transformation F has two unique multiplicative decompositions, i.e.,

$$\underline{F} = \underline{R}\underline{U} = \underline{V}\underline{R} \tag{6}$$

where $\underline{R}$ is orthogonal and $\underline{U}$, $\underline{V}$ is symmetric positive definite.

One can define the right *Cauchy–Green* tensor as follows[21,22]:

$$\underline{C} = \underline{U}^2 = \underline{F}^T\underline{F} \Leftrightarrow C_{\alpha\beta} = x_{,\alpha}^k x_{,\beta}^m g_{km} \tag{7}$$

and the left *Cauchy–Green* tensor of the deformation is

$$\underline{B} = \underline{V}^2 = \underline{F}^T\underline{F} \Leftrightarrow B^{km} = x_{,\alpha}^k x_{,\beta}^m g^{\alpha\beta} \tag{8}$$

The eigenvalues of $\underline{U}$ and $\underline{V}$ are the principal stretches of the deformation. The eigenvalues of $\underline{C}$ and $\underline{B}$ are the squares of these principal stretches. Also, the principal invariance $I_k(\underline{A})$ of a tensor value $\underline{A}$ are defined as the coefficients of the following polynomial:

$$\det(\lambda\underline{1} + \underline{A}) = \sum_{m=0}^{n} I_m(\underline{A})\lambda^{n-m} \tag{9}$$

where $I_0(\underline{A}) = 0$.

If the principal stretches are denoted by λ_i, then the invariance of $\underline{C}$ and $\underline{B}$ are given by

$$I_{\underline{C}} = I_{\underline{B}} = \sum \lambda_i^2$$

$$II_{\underline{C}} = II_{\underline{B}} = \sum \lambda_i^2\lambda_j^2(i \neq j)$$

$$III_{\underline{C}} = III_{\underline{B}} = \prod \lambda_i^2 \tag{10}$$

Cauchy's first law of motion is given by

$$T_{Rk;\alpha}^{\alpha} + \rho_R b_k = \rho_R x_k'' \tag{11}$$

where ";" denote the *covariant* derivative.

EXPLICIT FORMS OF STRAIN ENERGY FUNCTION AND STRESS TENSOR FOR ISOTROPIC HYPERELASTIC INCOMPRESSIBLE MATERIALS

An elastic material for which a strain energy function exists is called a Green elastic or hyperelastic solid. For such materials, the following relation gives the rate of working of stresses on the body:[21,22]

$$\frac{dW}{dt} = tr\left\{\frac{\partial W}{\partial \underline{F}}\frac{dF}{dt}\right\} \tag{12}$$

and the work done on the path of deformation is given by

$$W(\underline{F}) = W(\underline{F_0}) + \int_C \frac{dW}{dt}\, d\underline{F} \qquad (13)$$

where C defines a path along the space of deformation gradient.

According to fundamental theorem on hyperelastic materials, if the strain energy function of a hyperelastic material has a strict minimum, the material is a simple solid, and any minimizing configuration is a natural state. The strain energy function for such materials can be expressed as a function of a certain number of scalars $I_1, I_2, \ldots, I_n$ that are called invariants, i.e.,

$$W = W(\underline{C}) = \tilde{W}(\underline{C}) = W(I_1, \ldots, I_n) \quad (14)$$

A hyperelastic material is isotropic if and only if its strain energy function $W(\underline{U}) = \tilde{W}(\underline{C})$ where $\underline{U}$ and $\underline{C}$ are taken relative to an undistorted state, is an isotropic function, i.e., an orthogonal invariant. For an isotropic material the strain energy may be expressed as an isotropic function of the right stretch tensor $\underline{U}$, or the left stretch tensor $\underline{V}$, or the right Cauchy–Green tensor $\underline{C} = \underline{F}^T\underline{F}$ or the left Cauchy–Green tensor $\underline{B} = \underline{F}\underline{F}^T$, i.e.,

$$W(\underline{F}) = W(\underline{U}) = W(\underline{V}) = \tilde{W}(\underline{C}) = \tilde{W}(\underline{B}) \quad (15)$$

The strain energy $W(\underline{V})$ for isotropic materials is an invariant and may expressed as a symmetric function of the principal stretches λ_i:

$$W(\underline{V}) = W(\lambda_1, \lambda_2, \lambda_3) \qquad (16)$$

One can also consider an orthogonal basis of proper vectors of the left stretch tensor $\underline{V}$. These vectors define the *principal axes* of strain in the deformed state, which coincide with the principal axes of the stress.

The stress tensor T is given by

$$\underline{T} = -p\underline{1} + 2\rho\, \frac{\partial \tilde{W}}{\partial I_{\underline{B}}}\, \underline{B} - 2\rho\, \frac{\partial \tilde{W}}{\partial II_{\underline{B}}}\, \underline{B}^{-1} \quad (17)$$

which plays an important role in finite elasticity.

The principal stresses t_α, which correspond to principal stretches λ_α, are related by the following relation:

$$t_\alpha = \rho\lambda_\alpha \frac{\partial W}{\partial \lambda_\alpha} \ (\alpha = 1, 2, 3) \qquad (18)$$

where the parameter p can be determined from the boundary conditions of the problem under investigation.

LOGARITHMIC STRAIN MEASURE AND ITS INVARIANCES

As pointed out in the sectionon kinematics, the three eigenvalues of the left stretch tensor $\underline{V}$ are denoted by λ_1, λ_2, and λ_3 and their corresponding orthonormal eigenvectors by n_1, n_2, n_3. The general Eulerian strain measure can be defined as follows[22]:

$$\underline{e} = f(\underline{V}) = \sum_{i=1}^{3} f(\lambda_i)\underline{n}_i \otimes \underline{n}_i \qquad (19)$$

where f is a scale smooth monotone function.

One can obtain all commonly known Eulerian strain measures by taking the scale function $f(x)$ as certain particular forms. Assuming that $f(x) = \ln(x)$, the Eulerian logarithmic strain measure is given by

$$\ln(\underline{V}) = \sum_{i=1}^{3} \ln(\lambda_i)\underline{n}_i \otimes \underline{n}_i \qquad (20)$$

In general, it is difficult to calculate the logarithmic strain measure and its rates using the last equation. As noticed by Truesdell,[22] the logarithmic strain measure was restricted to certain particular deformations, such as simple tension, and was more general for deformations whose principal stretch axes n_i are invariant or using the direct or explicit basis-free expressions for $\ln(\underline{V})$ and its rates.

According to Hamilton principal,[21,22] every matrix satisfies its own characteristic equation, i.e.,

$$e_\alpha^3 - I_e e_\alpha^2 + II_e e_\alpha - III_e = 0 \qquad (21)$$

where the first invariance of the logarithmic strain is given by

$$I_e = tr(e) = \sum_{i=1}^{3} e_i, \ I_{ee} = tr(e^2) = \sum_{i=1}^{3} e_i^2,$$

$$I_{eee} = tr(e^3) = \sum_{i=1}^{3} e_i^3 \quad (22)$$

and the second is

Table I Various Forms of Strain Energy Function for Hyperelastic Solids

$W(I_c, II_c) = C_1(I_C - 3) + C_2(II_C - 3)$ — Mooney–Rivlin[9–11]

$W(I_c) = C_1(I_C - 3)$ — Neo-Hookean[21]

$W(I_C, II_C) = C_1(I_C - 3) + C_2\ln(II_C/3)$ — Gent–Thomas[25]

$\dfrac{\partial W}{\partial I_C} = G\exp[\alpha(I_C - 3)^2]$ and $\dfrac{\partial W}{\partial II_c} = G\beta/II_c$ — Hart–Smith[26]

$W(\lambda_1, \lambda_2, \lambda_3) = 2\mu \displaystyle\sum_{i=1}^{3} \lambda_i[\ln(\lambda_i) - 1]$ — K. Valanis and R. Landel[13]

$W = \dfrac{2G}{n} I_E + BI_E^m$

  where $I_E = \dfrac{1}{n} \displaystyle\sum_{i=1}^{3} (\lambda_i^n - 1)$ — P. J. Blatz et al.[26]

$W(\lambda_1, \lambda_2, \lambda_3) = \displaystyle\sum_{n=1}^{M} \dfrac{\mu_n}{\alpha_n} \sum_{i=1}^{3} (\lambda_i^{\alpha n} - 3)$ — R. W. Ogden[14,21]

$W = nkT\left[\dfrac{\lambda}{\sqrt{n}}\beta + \ln\left(\dfrac{\beta}{\sinh\beta}\right)\right] - W_0$

  where $\beta = L^{-1}\left(\dfrac{\lambda}{\sqrt{n}}\right), L(\beta) = \cot\beta - 1/\beta$ — P. D. Wu and E. Giessen[17]

$W = nk\Theta\left[\dfrac{1}{2}(I_C - 3) + \dfrac{1}{20N}(I_1^2 - 9) + \dfrac{11}{1050N^2}(I_1^3 - 27)\right]$ — Arruda and Boyce[16]

$\qquad\qquad + nk\Theta\left[\dfrac{19}{7000N^3}(I_1^4 - 81) + \dfrac{519}{673750N^2}(I_1^5 - 243)\right] + \cdots$

$W = -\dfrac{E}{6} J_m\ln(1 - J_1/J_m)$

  where $J_1 = \displaystyle\sum_{i=1}^{3} \lambda_i^3 - 3$ and $J_m = J_{1|\max}$ — A. Gent[15]

$W(\lambda) = \dfrac{E}{\alpha}\left\{\dfrac{(1+e)^{\alpha+1} - 1}{\alpha + 1} - e\right\}$

$e = \ln(\lambda)$ — P. A. Kakavas[18]

$II_e = \dfrac{1}{2}(I_e^2 - I_{ee}) = \dfrac{1}{2}\displaystyle\prod_{i,j=1}^{3} e_ie_j(\,\forall\, i \neq j)$

$\qquad\qquad = e_1e_2 + e_2e_3 + e_1e_3$ (23)

$III_e = \dfrac{1}{6}(I_e^3 - 3I_eI_{ee} + 2I_{eee}) = e_1e_2e_3$ (24)

Reference 23 gives the solution of eq. (21), i.e.,

$e_\alpha = \dfrac{1}{3}\left(I_e + 2\sqrt{I_e^2 - 3II_e}\,\cos\left[\dfrac{M + 2\pi(\alpha - 1)}{3}\right]\right)$ (25)

where

$M = \arccos\left\{\dfrac{2I_e^3 - 9I_eII_e + 27III_e}{2(I_e^2 - 3II_e)}\right\}$ (26)

The strain energy function W, can be defined as follows:

$W(e_1, e_2, e_3) = W(I_e, I_{ee}, I_{eee}) = W(I_e, II_e, III_e)$ (27)

  Choosing a logarithmic strain measure (the so-called Hencky's strain), one can obtain the first strain invariant, i.e.,

$$I_e = \sum_i e_i = \sum_i \ln(\lambda_i) = \ln\left(\prod_i \lambda_i\right) = \ln(J) \quad (28)$$

For incompressible materials, $J = 1$ and hence eq. (28) leads to

$$I_e = 0 \Rightarrow e_3 = -(e_1 + e_2) \quad (29)$$

and the strain energy function W can be written as

$$W = W(I_{ee}, I_{eee}) - pI_e \quad (30)$$

In addition, the second and the third invariance are

$$\frac{I_{ee}}{2} = (e_1 + e_2)^2 - e_1 e_2 \quad (31a)$$

$$\frac{I_{eee}}{3} = -e_1 e_2(e_1 + e_2) \quad (31b)$$

and if one lets the summation and the product of the first and the second components of the logarithmic strains to be written as

$$I = e_1 + e_2 \text{ and } II = e_1 e_2 \quad (32)$$

then eqs .(31) can be written as

$$\frac{I_{ee}}{2} = I^2 - II \quad (33a)$$

$$\frac{I_{eee}}{3} = -I\left(I^2 - \frac{I_{ee}}{2}\right) \quad (33b)$$

Equating the derivative of I_{eee} as a function of I to zero, i.e., $dI_{eee}/dI = 0$, one gets the following relation:

$$I^* = \pm(3I_{ee}/2)^{1/2} \quad (34)$$

and replacing into eq. (33b) yields the extreme values of I_{eee}, i.e.,

$$\left.\frac{I_{eee}}{6}\right|_{\text{extreme}} = \pm\left(\frac{I_{ee}}{6}\right)^{3/2} \quad (35)$$

In the special case of *simple tension* (ST) and/or equibiaxial compression (EBC) where the princi-
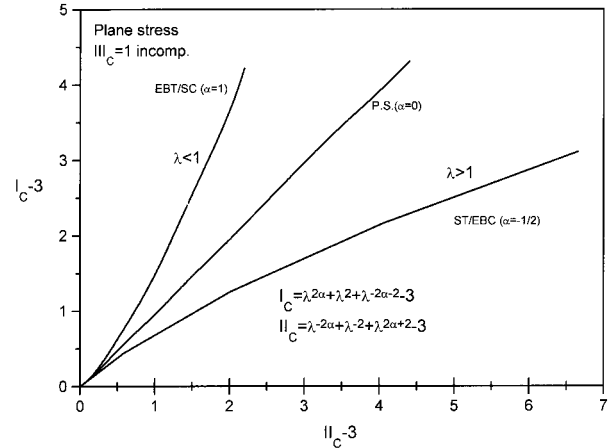

Figure 1 The second invariance of strain II_c as a function of the first invariant I_c for various type of deformation ($\alpha = -1/2 \Rightarrow$ simple tension and/or equibiaxial compression, $\alpha = 1 \Rightarrow$ simple compression and/or equibiaxial tension, $\alpha = 0 \Rightarrow$ pure shear).

ple stretches λ_α are equal to $\lambda, \lambda^{-1/2}, \lambda^{-1/2}$, the logarithmic strains are $e_\alpha = (e, -e/2, -e/2)$, where $e_\alpha = \ln(\lambda_\alpha)$. The values of I_{ee} and I_{eee} for this type of deformation are equal to $3e^2/2$ and $3e^3/4$, respectively. Hence, for ST deformation the following relation holds:

$$I_{eee}|_{\text{extreme}} = +6\left(\frac{I_{ee}}{6}\right)^{3/2} \text{ (ST/EBC)} \quad (36a)$$

For EBT and/or *simple compression*, the principle stretches λ_α are equal to $\lambda, \lambda, \lambda^{-2}$, and the logarithmic strains are $e_\alpha = e, e, -2e)$, where again $e = \ln(\lambda)$, and for these types of deformations one gets

$$I_{eee}|_{\text{extreme}} = -6\left(\frac{I_{ee}}{6}\right)^{3/2} \text{ (SC/EBT)} \quad (36b)$$

See Figure 1. For *pure shear* (PS), the principle stretches are $\lambda, 1, \lambda^{-1}$, i.e., $e_\alpha = (e, 0, -e)$ and

$$I_{eee}|_{\text{extreme}} = 0 \text{ (PS)} \quad (37)$$

For incompressible materials, a plot of I_{eee} as a function of I_{ee} for all possible deformations is shown in Figure 2.

STRAIN ENERGY FUNCTION BASED ON THE LOGARITHMIC STRAIN MEASURE

Results and Discussion

The available domain of the strain energy function $W = W(I_{ee}, I_{ee})$ is shown in Figure 2, which can be represented in the form
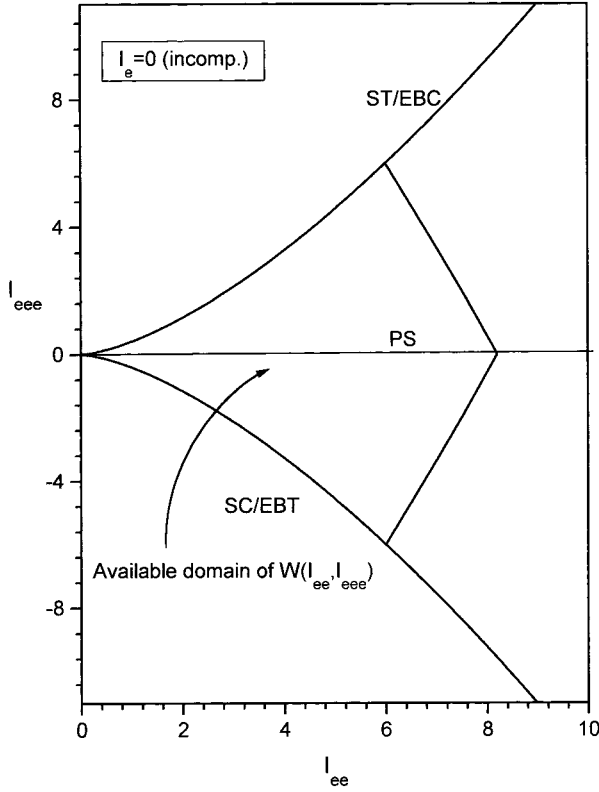
Figure 2 Plot of the logarithmic third invariant of strain I_{eee} as a function of the second one I_{ee} and the available domain of the strain energy function W.

$$\tilde{W}(I_{ee}, I_{eee}) = \sum_{i,j=0}^{\infty} c_{ij} a^i b^j (0 < a < 1, |b| < 1) \quad (38)$$

where $a = I_{ee}/I_{ee,\text{max}}$, $b = I_{eee}/I_{eee,\text{ max}}$, and the invariance $I_{eee,\text{ max}}, I_{ee,\text{max}}$ are related via eqs. (35).

Simple Tension

For this type of deformation, the principal stretches are equal to $(\lambda, \lambda^{-1/2}, \lambda^{-1/2})$ the components of the logarithmic strain are given by $(e, -e/2, -e/2)$. Using eq. (31), one can easily prove that the logarithmic strain e for this type of deformation is proportional to square root of the second invariant of logarithmic strain I_{ee}, i.e.,

$$e = \sqrt{2I_{ee}/3} \quad (39a)$$

and to qubic root of the third invariant of the strain I_{eee}, i.e.,

$$e = \sqrt[3]{4I_{eee}/3} \quad (39b)$$

Figure 3(a) shows the experimental data published by Treloar[7,12] of the *true stress t* as a function of the logarithmic strain. The experimental points were fitted with a polynomial of fifth degree and the derived curve was plotted on the same diagram. The strain energy function W was derived from the true stress W as follows:

$$t = \lambda \frac{\partial W}{\partial \lambda} \Leftrightarrow W(e) = \int_{e=0}^{e} t(e)\, de \quad (40)$$

where the variable e represents the logarithmic strain. The area underneath the curve t versus e represents the stored strain energy function W and the plot of W_{ST} as a function of I_{ee} and I_{eee} is shown in Figure 3(b). One can easily observe that for simple tension experiments the strain energy function W has the following form:

$$\tilde{W}_{\text{ST}}(I_{ee}, I_{eee}) = h\mathbf{I_{eee}} + g\mathbf{I_{ee}} + k\mathbf{I_{ee}^{3/2}} \quad (41)$$

where the constants h, g, and k are determined by fitting with the experimental curve. One using eqs. (40) and (41) may write the strain energy function as follows:

$$\tilde{W}_{\text{ST}}(e) = \frac{3}{2}(ge^2 + me^3) \quad (42)$$

where $m = (h + \sqrt{6}k)/2$.

Therefore, the strain energy function for simple tension is reduced to two material parameters (g and m). For $e = 1$, eq. (42) yields that $\tilde{W}(e) = 0$ as it should be in the undeformed state.

Assuming that the material is *incompressible*, then the true stress is given by[21,22]

$$t(e) = \frac{d\tilde{W}}{de_1} - \frac{d\tilde{W}}{de_2} = 3\frac{d\tilde{W}}{de} \; \forall\, \mathbf{e} \equiv \mathbf{e}_1 \quad (43a)$$

Replacing the derivative of $\tilde{W}$ upon e from eq. (42) gives the following relation for the true stress:

$$t(e) = 9ge + \frac{27}{2}me^2 \quad (43b)$$

For small strain, the logarithmic strain e is almost equal to true strain ε [i.e., $e = \ln(\lambda) = \ln(1 + \varepsilon) \approx \varepsilon$] and eq. (43b) reduces to Hook's law, i.e.,
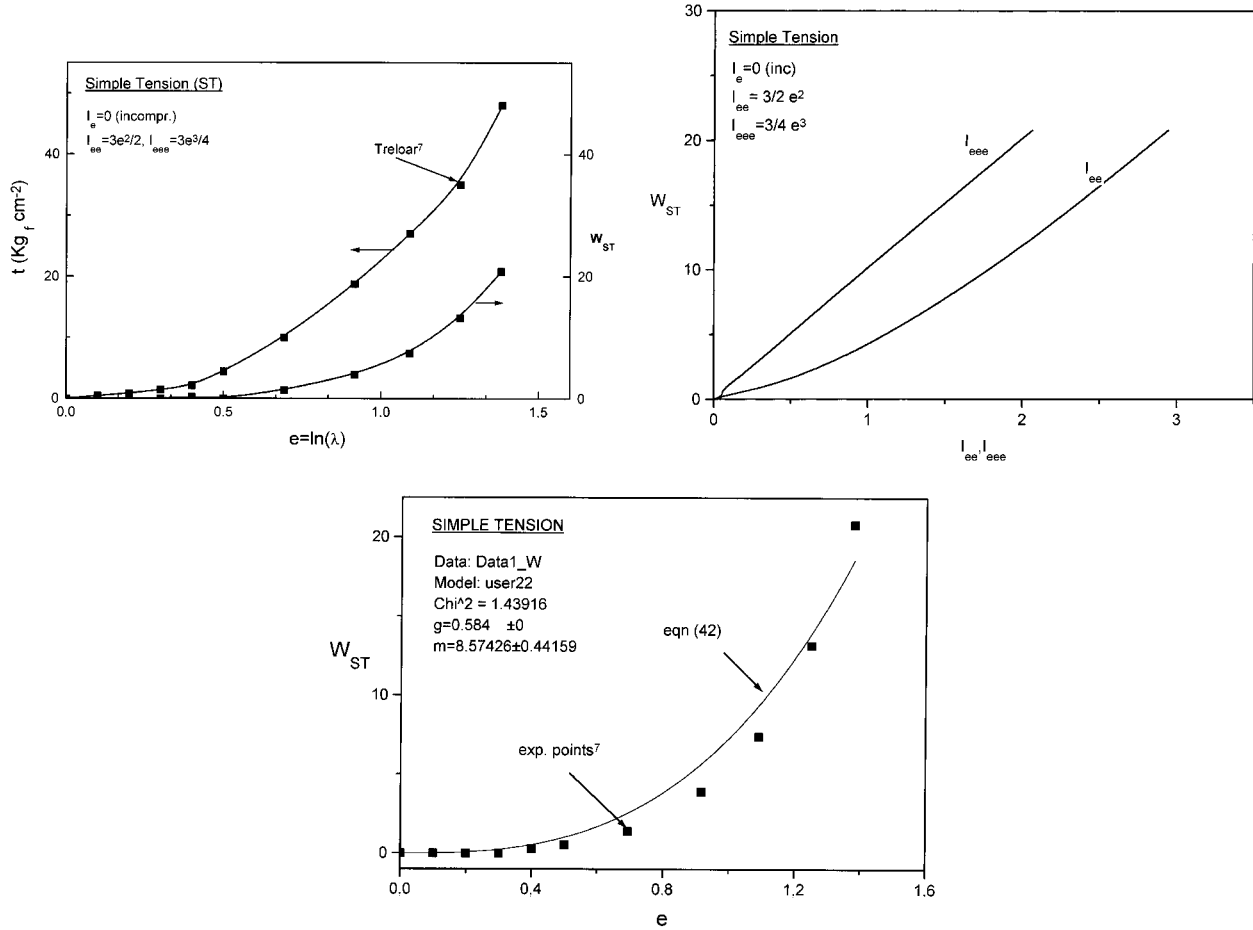
Figure 3 (a) True stress as a function of logarithmic strain for simple tension experimental data. Numerical integration of t (e) yields the form of the strain energy function. (b) Plot of W for simple tension experiments versus I_{ee} and I_{eee}. (c) Fitting of the strain energy function from simple tension data (ref. 7) with eq. (42).

$$t = 9ge \Rightarrow g = \frac{E}{9} \qquad (44)$$

where E denotes the Young's modulus of the material.

The parameter g is directly related to the modulus of elasticity of the material; therefore, it has a clear physical meaning. Figure 3(a) shows the experimental data for simple tension (ref. 7) that are fitted with a polynomial of fifth degree, and utilizing eq. (40) one can extract the values of the strain energy function. The results of W versus e for the Treloar data[7] are shown in the same plot. The strain energy function as a function of the invariant of logarithmic strain I_{ee} and I_{eee} are shown in Figure 3(b). The modulus of elasticity of the material used by Treloar[7] is equal to 5.256 kg$_f$/cm^2, as estimated

from Figure 3(a). Hence, eq. (44) yields a value of g equal to 0.584 kg$_f$/cm^2 . This value of g must fit all the possible forms of deformation, e.g., simple tension, equibiaxial tension and/or simple compression, and pure shear. Figure 3(c) shows the plot of the strain energy function as a function of logarithmic strain for simple tension data [see Fig. 3(a)]. The solid line represents the nonlinear fitting (with Origin 4.0) of the experimental data, derived from the true stress vs logarithmic strain, using eq. (42). The material parameter g was kept constant equal to 0.584 kg$_f$/cm^2 and the derived value of m was equal to 8.574 kg$_f$/cm^2. That is, the data were fitted perfectly well with only one parameter. Figure 3(d) shows the true stress versus e of Treloar data[7] for simple tension, and the solid line describes the fitting line with eq. (43b). For
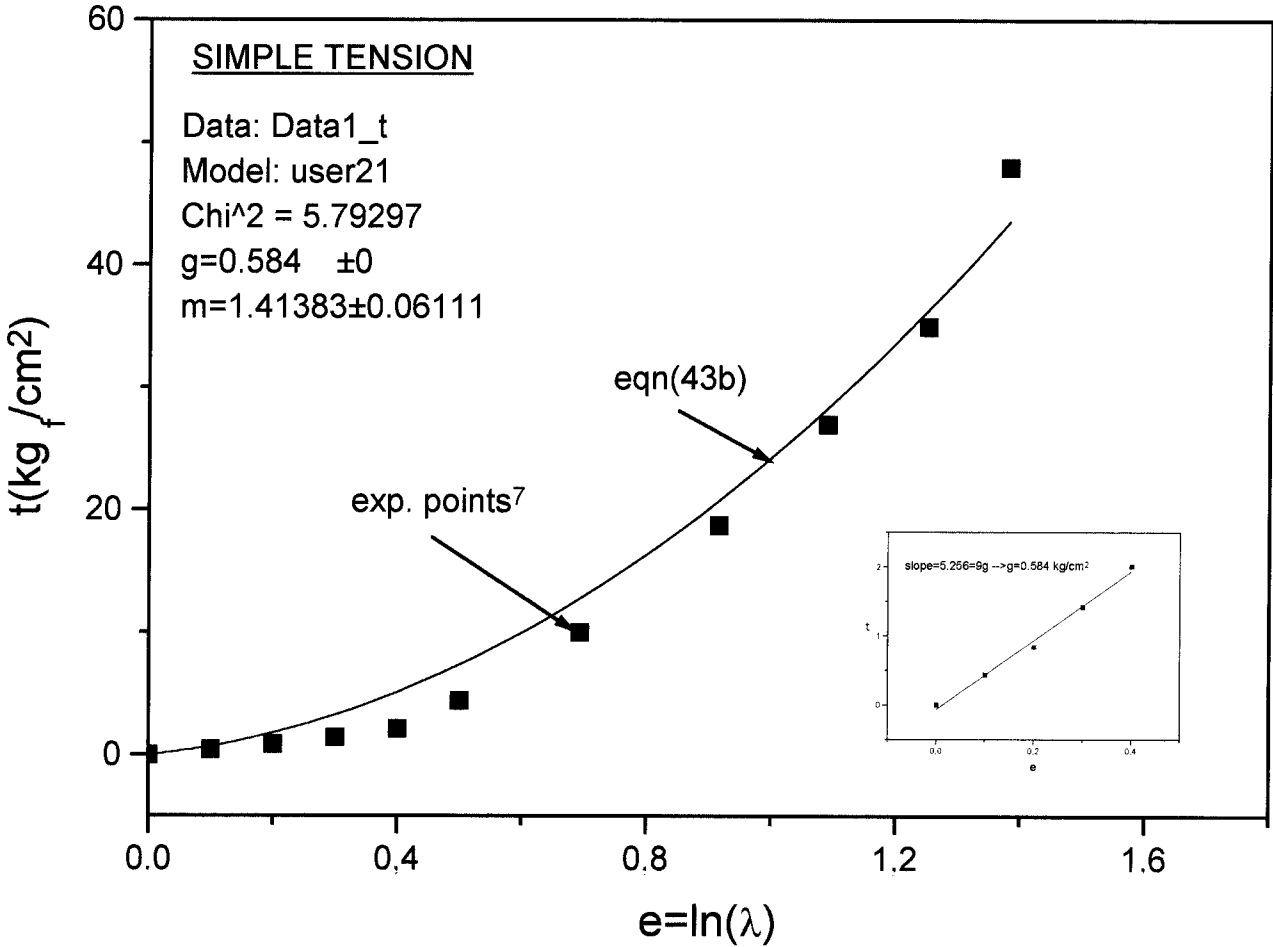
Figure 3(d) Fitting the true stress data for simple tension (ref. 7) with eq. (43b).

the parameter g was assigned the same value (i.e., $E/9$) as before, and the parameter m was estimated to be 1.414 kg_f/cm^2, which once more indicates that only one parameter was enough to fit the data.

Equibiaxial Tension and /or Simple Compression

The principal stretches for such deformations are $(\lambda, \lambda, \lambda^{-2})$ and the logarithmic strains are given by $(e, e, -2e)$. (where $e \equiv e_1 = e_2$), which are related to the second and third logarithmic invariance of strain by the relations

$$e = \sqrt{I_{ee}/6} \quad \text{and} \quad e = \sqrt[3]{-I_{eee}/6} \qquad (45a)$$

or equivalently,

$$I_{ee} = 6e^2, \ I_{eee} = -6e^3 \qquad (45b)$$

Replacing eq. (45b) into eq. (41) yields the form of the strain energy function for equibiaxial tests in terms of the logarithmic strain e, i.e.,

$$\tilde{W}_{\mathrm{EBT/SC}}(e) = 6[ge^2 + m'e^3] \qquad (46)$$

where $m' = -h + \sqrt{6}k$.

Considering that the material is incompressible, the true stresses in directions 1 and 2 are equal, and the following relation can determine them:

$$t_1 = t_2 = \frac{d\tilde{W}}{de_1} - \frac{d\tilde{W}}{de_2} = \frac{3}{2}\frac{d\tilde{W}}{de} \qquad (47a)$$

Substitution of eq. (46) into (47a) yields the form of the true stress, i.e.,
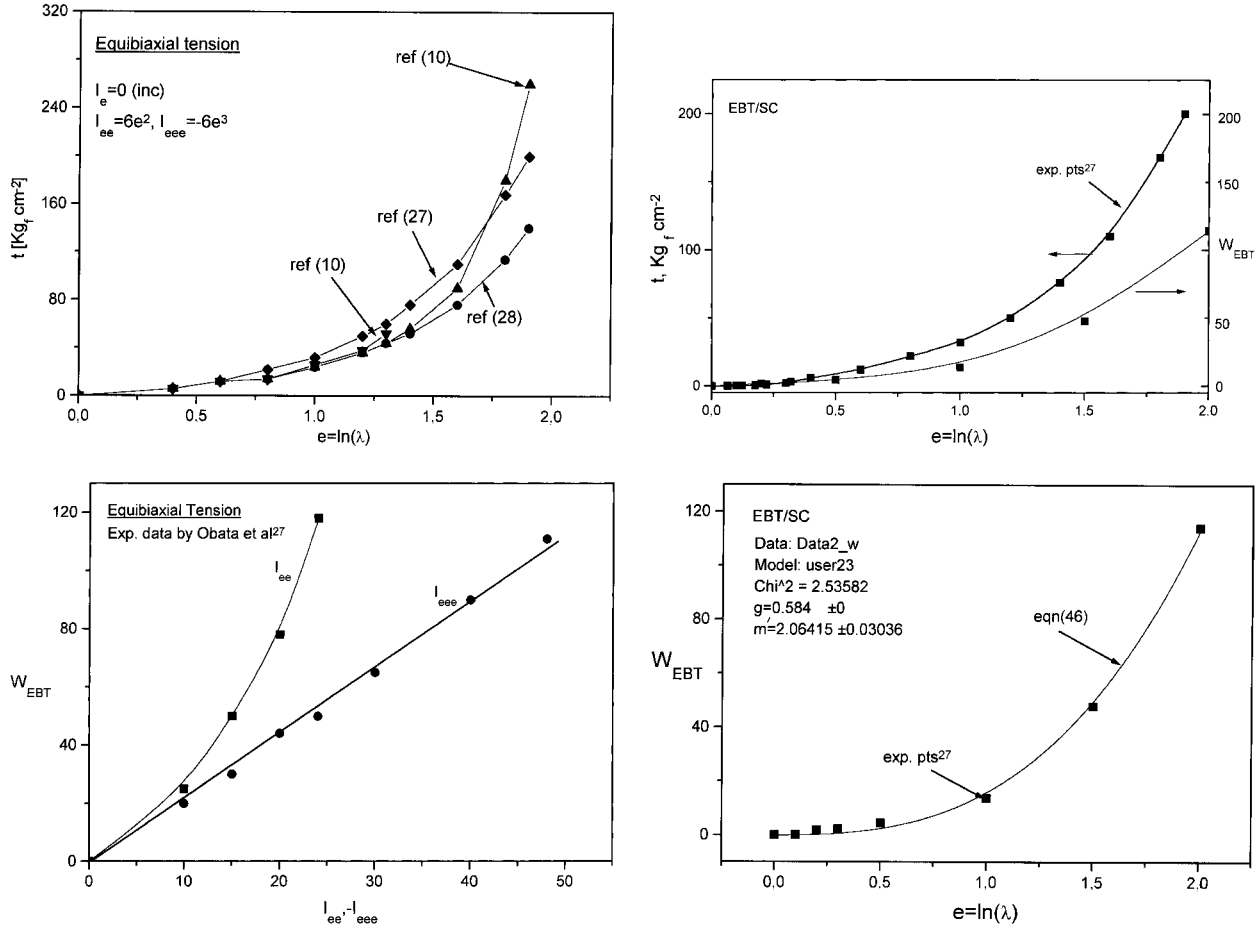
$$t(e) = 18ge + 27m'e^2 \qquad (47b)$$

Figure 4 (a) Equibiaxial data of true stress as a function of the logarithmic strain. (b) Plot of the true stress versus $e = \ln(\lambda)$ (ref. 27) and the derived form of the strain energy function W_{EBT}. (c) Plot of the strain energy function for equibiaxial data vs the second I_{ee} and third I_{eee} logarithmic invariant. (d) Fitting of the extracted strain energy function with eq. (46).

Figure 4(a) shows experimental data for rubber materials subjected to equibiaxial conditions.[9,10,27,28] For the present analysis, the data presented by Obata et al.[27] were used, which are shown in Figure 4(b). These data were fitted with a polynomial and their integration yields the experimental strain energy function, which is also presented in the same figure. A plot of the strain energy function as a function of the second and third invariance of logarithmic strain I_{ee} and I_{eee} is shown in Figure 4(c), which indicates that the strain energy function for equibiaxial conditions of a rubber material follows the form of eq. (41). The data of the strain energy function were fitted with eq. (46) using the nonlinear fitting algorithm based on Levenberg–Marquardt method that is written in the professional computer program Origin 4.0. The values of the derived material parameters were $g = 0.584$ $\mathrm{kg_f/cm^2}$ and $m' = 2.064$

$\mathrm{kg_f/cm^2}$, respectively. The fitting of the strain energy data is shown in Figure 4(d) with solid line. Notice that the fitting is obtained with the same value of parameter g, equal to one ninth of the value of the modulus of elasticity of the material derived from uniaxial data. The experimental data for the true stress as a function of the logarithmic strain were fitted with eq. (47b) using the same nonlinear algorithm [see Fig. 4(e)]. The fitting is shown with the solid curve on the figure and the derived parameters are $g = 0.584$ and $m' = 1.555$ $\mathrm{kg_f/cm^2}$, respectively. In the insertion, one observes that the fitting is much better for deformation up to $e = 1.2$, or equivalently, $\lambda = 3.32$ (i.e., strain 232%).

Pure Shear

For *pure shear* (PS) type of deformation in incompressible hyperelastic solids, for rubber like ma-
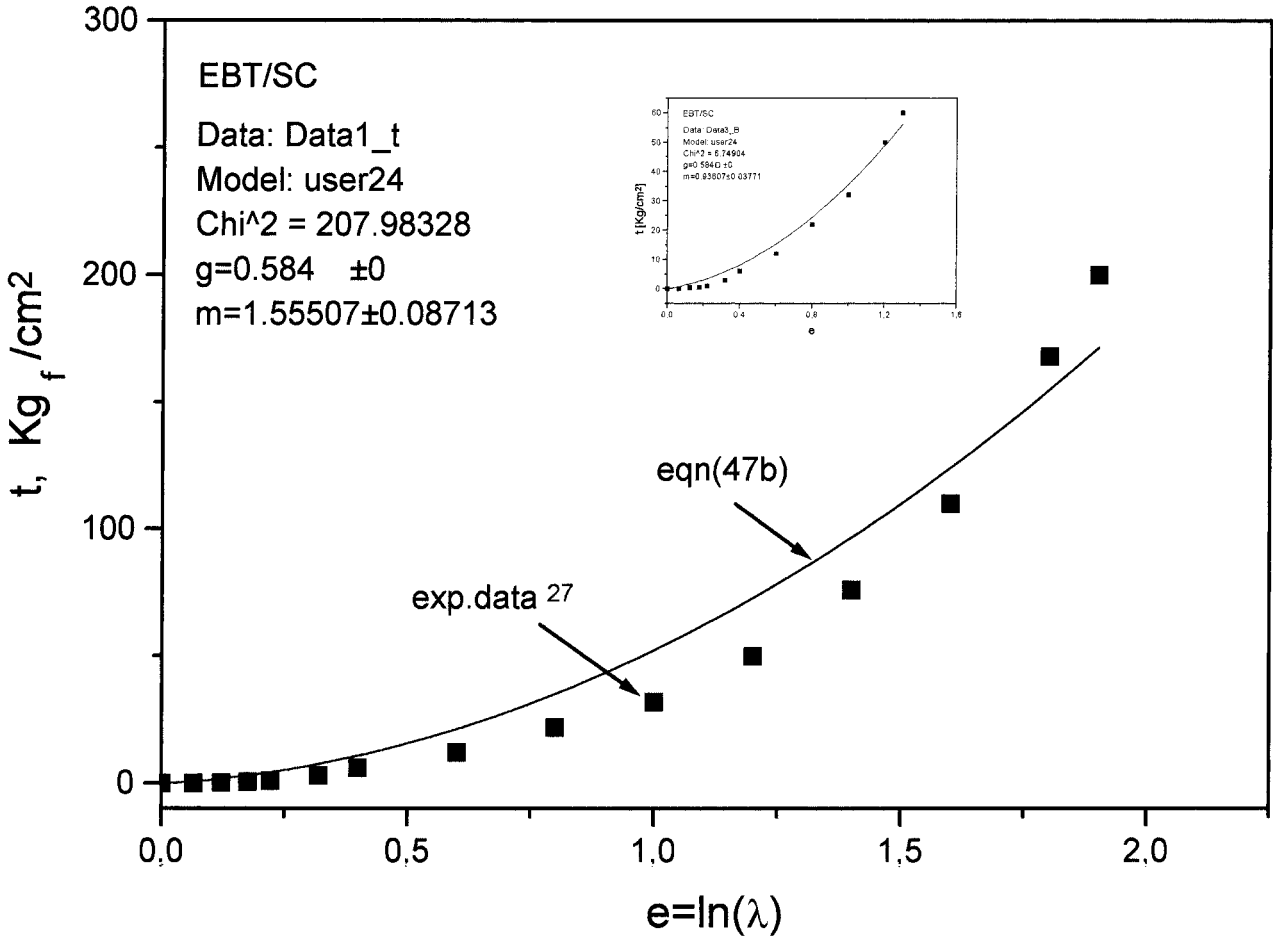
Figure 4(e)   Fitting the experimental true stress data (from ref. 27) with eq. (47b).

terials, the principal stretches are $(\lambda, 1, \lambda^{-1})$ and the components of the logarithmic strain are given by $(e, 0, -e)$. The corresponding invariance of the logarithmic strain are given by $I_e = 0$, $I_e = 2e^2$, and $I_{eee} = 0$, so that for such deformations the strain energy function depends only on the second invariant of logarithmic strain, i.e., $W_{PS} = W(I_{ee})$. Experimental data published by Rivlin[10] under pure shear conditions are shown in Figure 5(a). Fitting the data on the true stress with a polynomial of fifth degree and using eq. (40), one can compute the experimental values of the strain energy for pure shear. The resulting experimental data are plotted on the same Figure 5(a) that were fitted with a b-spline as the solid line shows it. The logarithmic strain invariance I_{ee} and I_{eee} are computed via eqs. (31) as a function of the logarithmic strain e, i.e.,

$$I_{ee} = 2e^2, \; I_{eee} = 0 \qquad (48)$$

The nonlinear dependence of W upon I_{ee} is shown in Figure 5(b). Using eq. (41), one can compute the

strain energy function as a function of the second logarithmic invariance I_{ee}, i.e.,

$$\tilde{W}_{PS}(I_{ee}) = gI_{ee} + kI_{ee}^{3/2} \Rightarrow \tilde{W}_{PS}(e) = 2(ge^2 + m''e^3)$$
$$(49)$$

where $m'' = \sqrt{2k}$.

Using the previously described nonlinear algorithm, one can estimate that the parameters g and m'' are equal to 0.584 and 10.619 kg$_f$/cm^2, respectively. Figure 5(c) shows the fitting of the experimental derived data for W from Rivlin's[10] published work with eq. (49).

The true stress for pure shear is given by

$$t(e) = 2\frac{d\tilde{W}}{de} = 8\left(ge + \frac{3}{2}m''e^2\right) \qquad (50)$$

Figure 5(d) shows the fitting of the experimental data with eq. (50) and the derived values of g and m'' were computed to be equal to 0.584 and 2.00 kg$_f$/cm^2, respectively.
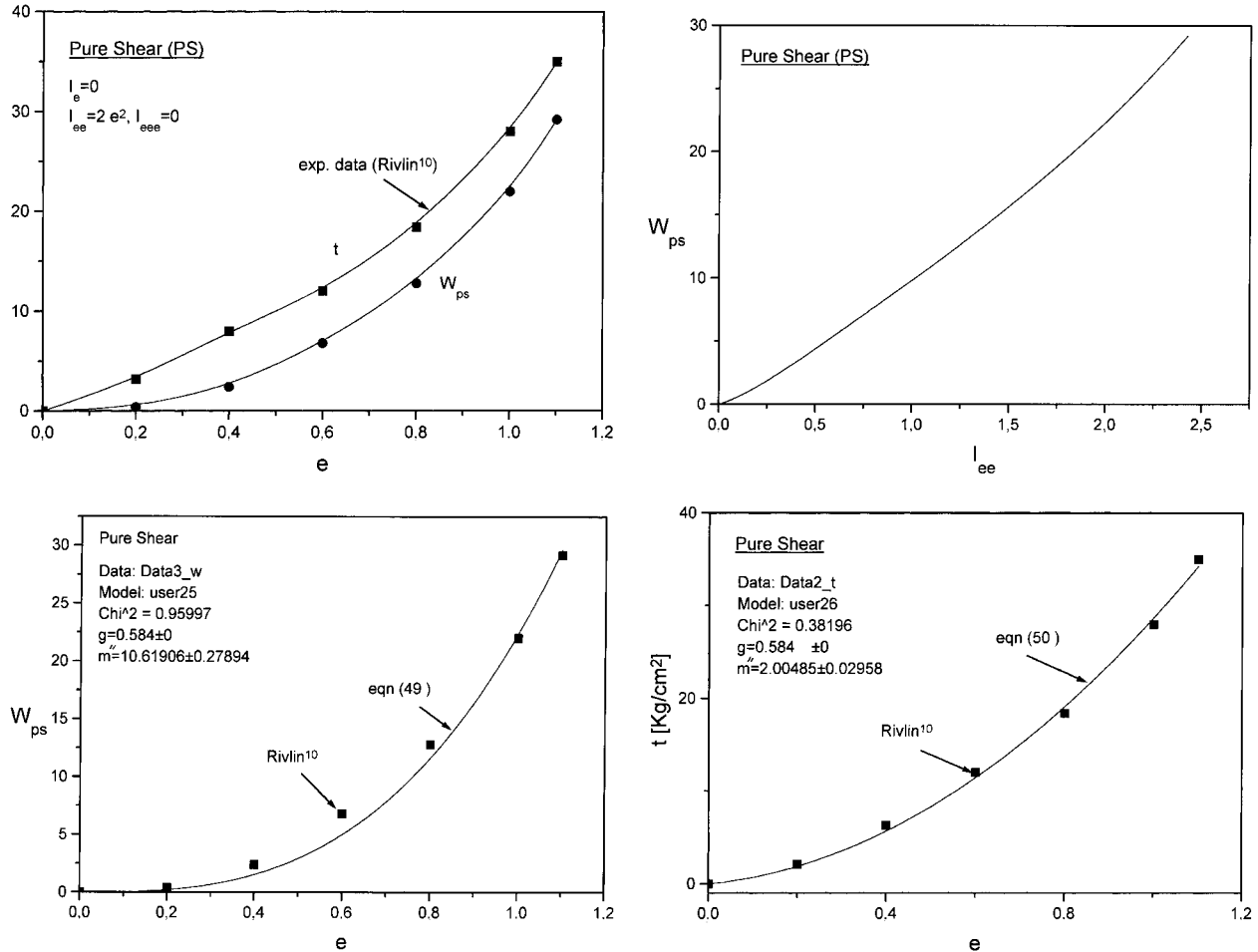
Figure 5 (a) Experimental data of true stress (ref. 10) under pure shear conditions and the derived strain energy function. (b) Strain energy function for pure shear data versus the I_{ee} logarithmic strain invariant. (c) Fitting of the derived W_{ps} from Rivlin data[10] with eq. (49). (d) Fitting the true stress for pure shear data (ref. 10) with eq. (50).

CONCLUSIONS

The present study shows that the strain energy function W for hyperelastic solids subjected to various types of deformation can be described by eq. (41), which involves three parameters. However, the three parameters are reduced to one, as previously shown for various types of deformation. The use of the logarithmic strain approach leads to a new form of the strain energy function based on two parameters. However, since one of the parameters is determined by the Young's modulus of the material, the model is reduced to one parameter. The computation of the unknown parameter m was obtained by fitting with the available experimental data from the literature.

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